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FILTER PROPERTIES OF LEAST SQUARES FITTED POLYNOMIALS

BODO KRUGER

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GODDARD SPACE FLIGHT CENTER
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FILTER PROPERTIES OF LEAST SQUARES FITTED POLYNOMIALS

Bodo Kruger

January, 1968

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SUMMARY

The use of least squares fitted polynomials (LSP) as high and low pass filters is analyzed in this paper.

A typical high pass filter application is noise analysis of tracking data. Equations are given for the estimation of degrees of freedom k needed in the LSP in order to "take out" the orbit so that the residuals (tracking data - LSP) represent the tracking data noise. As the LSP only contains k terms, the orbit is not completely "taken out" and what is left over of the orbit in the tracking data is called the modeling error δ. It is shown that the standard deviation of δ is proportional to T^k and $Y^{(k)}$

$$\sigma_\delta \sim T^k\, Y^{(k)}$$

where

T = the length of the data stretch

$Y^{(k)}$ = k^{th} derivative of the "orbit".

The modeling error is analyzed in some detail for a lunar orbit (see Figures 1 through 5) and it is shown that k = 7 is needed for both range and range rate data if the standard deviations σ_r and $\sigma_{\dot{r}}$ of the modeling errors have to satisfy

$$\sigma_r < 10 \text{ m}$$

$$\sigma_{\dot{r}} < 1 \text{cm/sec}$$

and if the data stretch is 1/4 revolution, starting at the earth-moon line.

A typical application of low pass filtering is data compression. Noisy data is substituted by one or more points of an LSP which has been fitted to the noisy data. The modeling error δ in this application introduces a bias in the compressed data points. It is shown that the commonly used midpoint of the LSP is not optimal with respect to the bias. For instance, for $k = 5$, the bias may be reduced by a factor of 1.53 if the optimal point is chosen instead of the midpoint.

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1.0 INTRODUCTION

In this report the filter properties of Least Squares Fitted Polynomials (LSP) are analyzed from the viewpoint of their use as high pass and low pass filters.

In the application as a high pass filter, noise is removed from data contaminated by noise by fitting a LSP to the contaminated data. The noise is contained in the residuals (contaminated data minus LSP) but some of the data "spills over" into the residuals. The data points may be thought of as being generated by a polynomial with infinitely many terms. Fitting a LSP with a finite number of terms to the data has to result in modeling errors. These modeling errors "spill over" into the residuals and it is therefore desirable to keep them small. The modeling errors can be made small by using a LSP with many degrees of freedom, i.e. many terms. On the other hand, the noise introduced by the computer increases with the degrees of freedom of the LSP. It is therefore desirable to determine the minimum number of degrees of freedom of the LSP with acceptable modeling errors. Expressions for the variance of the modeling errors are derived in Section 2 and their application to analysis of tracking data from a lunar orbit is demonstrated in Section 3.

In the application of the LSP as a low pass filter, noisy data is replaced by the LSP. The modeling error shows up as a "bias" and the noise on the data introduces a statistical uncertainty to the LSP. Expressions for both the bias and the statistical uncertainty are given in Section 4.

2.0 THE LEAST SQUARES FITTED POLYNOMIAL AS HIGH PASS FILTER

2.1 Data Without Noise

Assume that exact data points are generated by the polynomial

$$Y_j = b_0 + b_1 j + \ldots b_{k-1} j^{k-1} + b_k i^k + \ldots b_{n-1} j^{n-1} \qquad (2.1)$$

and that a polynomial

$$\bar{y}_i = a_0 + a_1 i + \ldots + a_{k-1} i^{k-1} \tag{2.2}$$

with k degrees of freedom is fitted to Y_j in the least squares sense. N equidistant values of Y_j are available and i and j stand for the integers from 1 to N. The sampling interval is normalized to 1, which can be done without loss of generality.

It is shown in Reference 1, Equation (10) that $\bar{y}_i$ may be written

$$\bar{y}_i = -\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} Y_j \tag{2.3}$$

where

$$|A| = \begin{vmatrix} A_0 & A_1 & \cdot & \cdot & \cdot & A_{k-1} \\ A_1 & A_2 & \cdot & \cdot & \cdot & A_k \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ A_{k-1} & A_k & \cdot & \cdot & \cdot & A_{2k-2} \end{vmatrix} \tag{2.4}$$

$$\begin{vmatrix} 0 & j \\ i & A \end{vmatrix} = \begin{vmatrix} 0 & 1 & j & \cdot & \cdot & \cdot & j^{k-1} \\ 1 & A_0 & A_1 & \cdot & \cdot & \cdot & A_{k-1} \\ i & A_1 & A_2 & \cdot & \cdot & \cdot & A_k \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ i^{k-1} & A_{k-1} & A_k & \cdot & \cdot & \cdot & A_{2k-2} \end{vmatrix} \tag{2.5}$$

$$A_v = \sum_{i=1}^{N} i^v \tag{2.6}$$

$\bar{y}_i$ can not be fitted exactly to Y_i if $k < n$.

The residuals or modeling error

$$\delta_i = Y_i - \bar{y}_i \tag{2.7}$$

may be written

$$\delta_i = \frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \left(b_0 + b_1 j + \ldots + b_{n-1} j^{n-1}\right) \tag{2.8}$$

$$+ \left(b_0 + b_1 i + \ldots + b_{n-1} i^{n-1}\right)$$

But for $m < k$ we obtain from Reference 1, Appendix E,

$$\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} b_m j^m = -b_m i^m \tag{2.9}$$

so that

$$\delta_i = \frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \left(b_k j^k \ldots b_{n-1} j^{n-1}\right) + \left(b_k i^k + \ldots + b_{n-1} i^{n-1}\right) \tag{2.10}$$

It is shown in Appendix A that this expression also may be written

$$\delta_i = \frac{(-1)^k}{|A|} \left\{ |i^k| \; b_k + \ldots + |i^{n-1}| \; b_{n-1} \right\} \tag{2.11}$$

where

$$
|i^{k+v}| = \begin{vmatrix} 1 & A_0 & A_1 & \cdots & A_{k-1} \\ i & A_1 & A_2 & \cdots & A_k \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ i^{k-1} & A_{k-1} & A_k & \cdots & A_{2k-2} \\ i^{k+v} & A_{k+v} & A_{k+v+1} & \cdots & A_{2k+v-1} \end{vmatrix} \tag{2.12}
$$

The variance $E\{\delta^2\}$ of the modeling error is

$$
E\{\delta^2\} = \frac{1}{N}\sum_{i=1}^{N} \delta i^2 = \frac{1}{N|A|^2} \sum_{p=0}^{n-k} \sum_{q=0}^{n-k} \sum_{i=1}^{N} |i^{k+p}| \cdot |i^{k+q}| \; b_{k+p} b_{k+q} \tag{2.13}
$$

In Appendix B it is shown that for large N

$$
\sum_{i=1}^{N} |i^{k+p}| \cdot |i^{k+q}| = \tag{2.14}
$$

$$
N^{2k+p+q+1} \frac{[(k+p)!]^2 \, [(k+q)!]^2}{p!\, q!\, (2k+p)!\, (2k+q)!\, (2k+p+q+1)} \cdot |A|^2
$$

Insertion in Equation (2.13) yields

$$E\{\delta\}^2 = N^{2k} \frac{[(k)!]^4}{[(2k)!]^2 (2k+1)} b_k^2 + N^{2k+2} \frac{[(k+1)!]^2}{[(2k+1)!] (2k+3)} b_{k+1}^2$$

$$+ N^{2k+4} \frac{[(k+2)!]^4}{4[(2k+2)!]^2 (2k+5)} b_{k+2}^2 + \ldots .$$

$$+ 2N^{2k+1} \frac{[(k)!]^2 [(k+1)!]^2}{(2k)!(2k+1)!(2k+2)} b_k b_{k+1} + 2N^{2k+2} \frac{[(k)!]^2 [(k+2)!]^2}{2(2k)!(2k+2)![2k+3]} b_k b_{k+2} + \ldots$$

$$+ 2N^{2k+3} \frac{[(k+1)!]^2 [(k+2)]^2}{2(2k+1)!(2k+2)!(2k+4)} b_{k+1} b_{k+2} + \ldots . \tag{2.15}$$

An alternate and useful expression for the variance is obtained by expressing the generating polynomial Y_j, Equation (2.1), in a Taylors series

$$Y_t = Y(0) + \frac{Y^{(1)}}{1!} t + \frac{Y^{(2)}}{2!} t^2 + \ldots + \frac{Y^{(n-1)}}{(n-1)!} t^{n-1} \tag{2.16}$$

If the sampling interval is h, then

$$t = jh \tag{2.17}$$

and thus by comparison with Equation (2.1)

$$b_v = \frac{Y^{(v)}}{v!} h^v \tag{2.18}$$

For large N we may write the interval of observation T as

$$T = N \cdot h \tag{2.19}$$

The variance is then

$$E\{\delta\}^2 = T^{2k} \frac{[(k!)]^2}{[(2k)!]^2(2k+1)} (Y^{(k)})^2 + T^{2k+2} \frac{[(k+1)!]^2}{[(2k+1)!]^2(2k+3)} (Y^{(k+1)})^2$$

$$+ T^{2k+4} \frac{[(k+2)!]^2}{4[(2k+2)!]^2(2k+5)} (Y^{(k+2)})^2 + \ldots .$$

$$+ 2T^{2k+1} \frac{(k)!(k+1)!}{(2k)!(2k+1)!(2k+2)} Y^{(k)}Y^{(k+1)} + 2T^{2k+2} \frac{(k)!(k+2)!}{2(2k)!(2k+2)!(2k+3)} Y^{(k)}Y^{(k+2)}$$

$$+ \ldots . + 2T^{2k+3} \frac{(k+1)!\,(k+2)!}{2(2k+1)!\,(2k+2)!\,(2k+4)} Y^{(k+1)}\,Y^{(k+2)} + \ldots . \quad (2.20)$$

In many cases only the first term needs to be considered and Equation (2.20) simplifies to

$$\sigma_\delta = T^k \frac{k!}{(2k)!\sqrt{2k+1}} |Y^{(k)}| \quad (2.21)$$

where

$$\sigma_\delta^2 = E\{\delta^2\}$$

If $Y(t)$ only contains even or odd terms k may be chosen so that $Y(k) = 0$ and

$$\sigma_\delta = T^{k+1} \frac{(k+1)!}{(2k+1)!\sqrt{2k+3}} |Y^{(k+1)}| \quad (2.22)$$

Sometimes it is desirable to consider a symmetric interval from -T to +T. The modeling error δ_i for the symmetric interval may be derived by the variable transformation

$$x = t + T \quad (2.23)$$

Thus if

$$-T \leq t \leq +T$$

then

$$0 \leq x \leq 2T$$

and the previously found equations for $E\{\delta^2\}$ are applicable to the x interval.

Substituting Equation (2.23) into (2.16) we obtain

$$Y = \sum_{r=0}^{n-1} \frac{Y^{(r)}}{r!} t^r = \sum_{r=0}^{n-1} \frac{Y^{(r)}}{r!} (x - T)^r$$

or

$$Y(x) = \sum_{m=0}^{n-1} \frac{\beta_m}{m!} x^m \tag{2.24}$$

where

$$\beta_m = \sum_{r=m}^{n-1} (-1)^{r-m} \frac{Y^{(r)}}{(r - m)!} T^{r-m} \tag{2.25}$$

Substitution Equation (2.25) into Equation (2.20) yields

$$E^*(\delta^2) = (2T)^{2k} \frac{(k!)^2}{[(2k)!]^2 (2k + 1)} (Y^{(k)})^2 \tag{2.26}$$

$$+ (2T)^{2k+2} \frac{(k!)^2}{[(2k)!]^2 (2k + 1)(2k + 3)} \left\{ \frac{(Y^{(k+1)})^2}{4(2k + 1)} + \frac{Y^{(k)}\, Y^{(k+2)}}{4} \right\} + \ldots$$

where $E^*(\delta^2)$ deviates the variance of the modeling error δ_i for the interval from $-T$ to $+T$.

From Equations (2.21) and (2.26) it is seen that σ_δ is proportional to the length of the interval raised to power k and proportional to the k^{th} derivative of Y if terms of higher order then k are negligible.

2.2 Data With Noise

Let y_j be the observed values of the true data points Y_j. If the data is contaminated with noise ϵ_j then

$$y_j = Y_j + \epsilon_j \tag{2.27}$$

The residuals are

$$v_i = y_i - \bar{y}_i$$

Hence from Equations (2.3) and (2.7)

$$v_i = \epsilon_i + \frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j + \delta_i \tag{2.28}$$

The variance $E\{v^2\}$ is thus

$$E\{v^2\} = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i^2 + \frac{1}{N} \sum_{i=1}^{N} \left[\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \right]^2 + \frac{1}{N} \sum_{i=1}^{N} \delta_i^2$$

$$+ \frac{2}{N} \sum_{i=1}^{N} \left[\frac{\epsilon_i}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \right] + \frac{2}{N} \sum_{i=1}^{N} \epsilon_i \delta_i + \frac{2}{|N|} \sum_{i=1}^{N} \left[\frac{\delta_i}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \right]$$

(2.29)

If the ϵ_i are uncorrelated stochastic variables with zero mean, then

$$\frac{1}{N} \Sigma \epsilon_i^2 = \sigma_\epsilon^2$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \epsilon_i \delta_i = 0 \tag{2.30}$$

because δ_i is a polynomial in i according to Equation (2.11)

$$\sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j$$

is then also a stochastic variable and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\delta_i}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j = 0 \tag{2.31}$$

In Reference 1, Appendix E, it is shown that

$$\sum_{i=1}^{N} \left[\frac{1}{|A|} \sum \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j \right]^2 = - \frac{1}{|A|} \begin{vmatrix} 0 & \Sigma_j \epsilon_j \\ \Sigma_j \epsilon_j & A \end{vmatrix} = k \, \sigma_\epsilon^2 \tag{2.32}$$

We also have

$$\sum_{i=1}^{N} \frac{\epsilon_i}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j = \frac{1}{|A|} \begin{vmatrix} 0 & \Sigma_j \epsilon_j \\ \Sigma_i \epsilon_i & A \end{vmatrix}$$

As it is immaterial whether the summation index is i or j, we thus have

$$\sum_{i=1}^{N} \frac{\epsilon_i}{|A|} \sum \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j = - k \sigma_\epsilon^2 \tag{2.33}$$

Using Equation (2.20) the final result is obtained

$$E \{v^2\} = \frac{N - k}{N} \sigma_\epsilon^2 + \left\{ \frac{k!}{(2k)! \sqrt{2k + 1}} Y^{(k)} T^k + \ldots \right\}^2 \tag{2.30}$$

The total variance is thus obtained by adding the noise variance to the model error variance. See also Reference 2.

3.0 ANALYSIS OF RANGE AND RANGE RATE DATA FROM A LUNAR ORBIT

In order to get a feel for the degree of polynomial needed for least squares fits to Lunar Orbiter data, the case of a circular lunar orbit is analyzed.

From Figure 1 we obtain, neglecting the earth rotation

$$r^2 = a^2 + b^2 - 2ab \cos \omega t \tag{3.1}$$

The application of Equations (2.20) through (2.26) requires a series expansion of the range r and the range rate $\dot{r}$. The easiest way seems to be to expand $\cos \omega t$ into a series and then to extract the root.

With

$$a = 3.844 \times 10^8 \text{ m}$$

$$b = 1.800 \times 10^6 \text{ m}$$

$$\mu = 4.903 \times 10^{-12} \text{ m}^3/\text{sec}^2$$

$$\omega^2 = 8.406 \times 10^{-7} \text{ (rad/sec)}^2$$

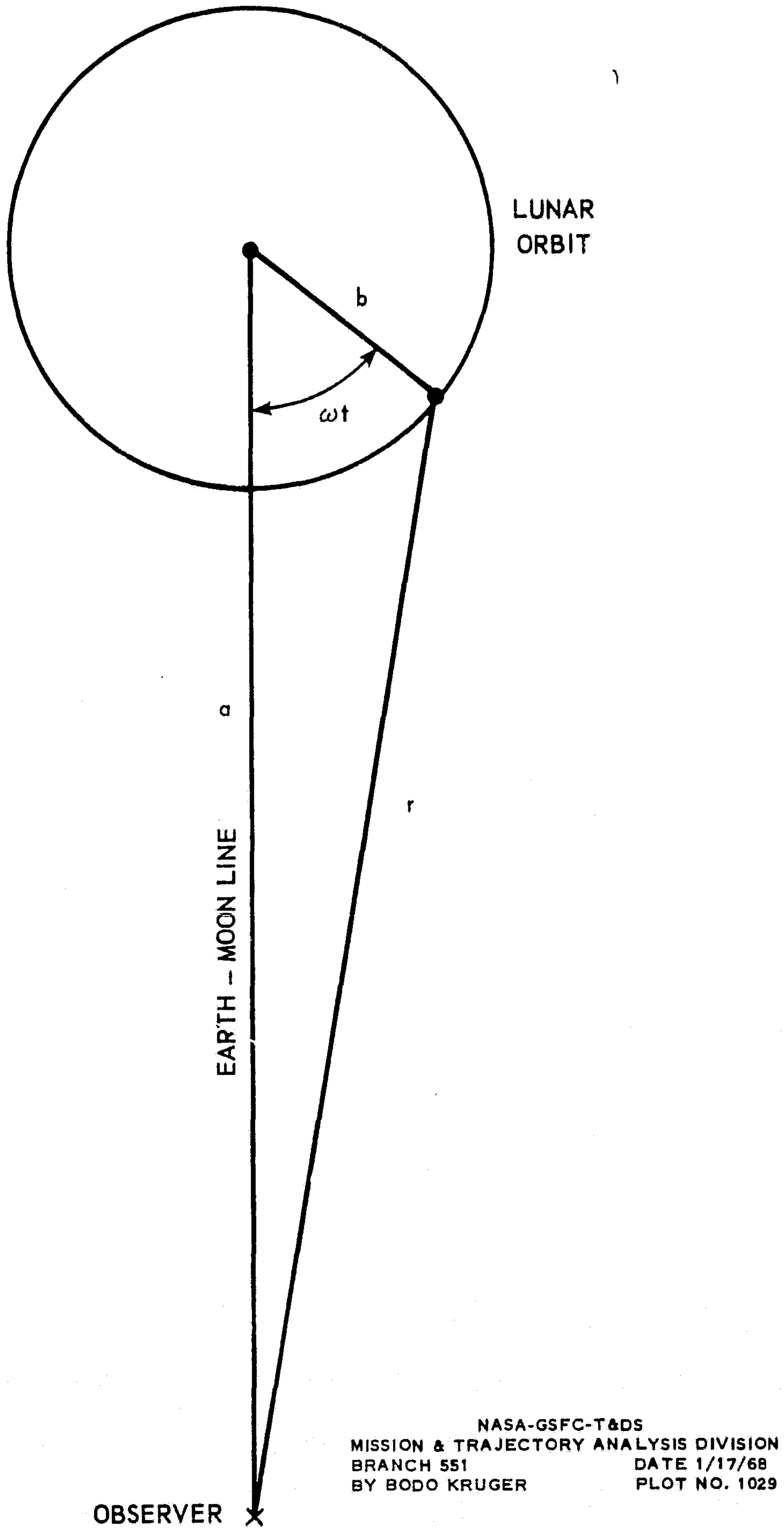

Figure 1. Lunar Orbit Geometry

the range r is

$$r = 3.826 \times 10^8 + \frac{1.524}{2!} t^2 - \frac{1.300 \times 10^{-6}}{4!} t^4 + \frac{1.156 \times 10^{-12}}{6!} t^6$$

$$- \frac{1.189 \times 10^{-18}}{8!} t^8 + \frac{1.796 \times 10^{-24}}{10!} t^{10} - \frac{4.72 \times 10^{-30}}{12!} t^{12} + \dots \tag{3.2}$$

where r is in meters and t in seconds. The range rate $\dot{r}$ in m/sec is

$$\dot{r} = 1.524 \cdot t - \frac{1.300 \times 10^{-6}}{3!} t^3 + \frac{1.156 \times 10^{-12}}{5!} t^5$$

$$- \frac{1.189 \times 10^{-18}}{7!} t^7 + \frac{1.796 \times 10^{-24}}{9!} t^9 - \frac{4.72 \times 10^{-30}}{11!} t^{11} + \dots \tag{3.3}$$

The standard deviations σ_r and $\sigma_{\dot{r}}$ of the modeling errors for range and range rate have been calculated for the lunar orbit using Equations (2.21), (2.22) and (2.26) both for the unsymmetric interval 0 to T and for the symmetric interval -T to +T. The results are summarized in Table 1 and are shown graphically in Figures 2 through 5. For the region of σ_r and $\sigma_{\dot{r}}$ shown in Figures 2 through 5, the one-term approximation leading to Equations (2.21), (2.22) and (2.26) is valid. For higher values of σ_r and $\sigma_{\dot{r}}$ the next term in the series expansions should be considered (see Section 2.1).

In Figures 4 and 5 are shown two points obtained by computer simulation. Using the geometry shown in Figure 1, 75 values for r and $\dot{r}$ were calculated for the time interval -2220 sec $\leq t \leq$ +2220 sec. A polynomial with k = 7 was fitted to the range data and a polynomial with k = 8 was fitted to the range rate data. The standard deviations of the residuals are in excellent agreement with the curves in Figures 4 and 5.

The time for one orbit is τ = 6,840 sec and for 1/4 of an orbit $\tau/4$ = 1,710 sec. Assume we want to fit a polynomial to 1/4 of an orbit with the modeling errors

$$\sigma_r < 10 \text{ m}$$
$$\sigma_{\dot{r}} < 1 \text{ cm/sec}$$

Table 1

The Standard Deviation of the Modeling Errors for Range and Range Rate for a Lunar Orbit

<table>
<tr><th rowspan="2">k</th><th colspan="2">Unsymmetric Interval
from 0 to T</th><th colspan="2">Symmetric Interval
from -T to +T</th></tr>
<tr><th>σ_r
m</th><th>$\sigma_{\dot{r}}$
m/sec</th><th>σ_r
m</th><th>$\sigma_{\dot{r}}$
m/sec</th></tr>
<tr><td>2</td><td>$5.86 \times 10^{-2} T^2$</td><td>$2.46 \times 10^{-8} T^3$</td><td></td><td rowspan="2">$3.28 \times 10^{-8} T^3$</td></tr>
<tr><td>3</td><td>$2.06 \times 10^{-9} T^4$</td><td>$4.10 \times 10^{-9} T^3$</td><td rowspan="2">$4.13 \times 10^{-9} T^4$</td></tr>
<tr><td>4</td><td>$2.58 \times 10^{-10} T^4$</td><td>$1.153 \times 10^{-16} T^5$</td><td rowspan="2">$3.69 \times 10^{-16} T^5$</td></tr>
<tr><td>5</td><td>$5.78 \times 10^{-18} T^6$</td><td>$1.153 \times 10^{-17} T^5$</td><td rowspan="2">$3.08 \times 10^{-17} T^6$</td></tr>
<tr><td>6</td><td>$4.82 \times 10^{-18} T^6$</td><td>$2.49 \times 10^{-25} T^7$</td><td rowspan="2">$2.27 \times 10^{-24} T^7$</td></tr>
<tr><td>7</td><td>$8.92 \times 10^{-27} T^8$</td><td>$1.775 \times 10^{-26} T^7$</td><td rowspan="2">$1.420 \times 10^{-25} T^8$</td></tr>
<tr><td>8</td><td>$5.56 \times 10^{-28} T^8$</td><td>$4.20 \times 10^{-34} T^9$</td><td rowspan="2">$1.195 \times 10^{-32} T^9$</td></tr>
<tr><td>9</td><td>$1.171 \times 10^{-35} T^{10}$</td><td>$2.35 \times 10^{-35} T^9$</td><td rowspan="2">$5.98 \times 10^{-34} T^{10}$</td></tr>
<tr><td>10</td><td></td><td></td><td rowspan="2">$6.47 \times 10^{-41} T^{11}$</td></tr>
<tr><td>11</td><td></td><td></td><td>$2.99 \times 10^{-42} T^{12}$</td></tr>
</table>

For the unsymmetric case, $0 \leq t \leq 1710$ sec we obtain from Figures 2 and 3 that a polynomial with $k = 7$ is needed for both range and range rate. If the interval is symmetric, $-855 \leq t \leq +855$ sec, we see from Figures 4 and 5 that a polynomial with $k = 7$ is needed for the range and $k = 6$ for range rate.

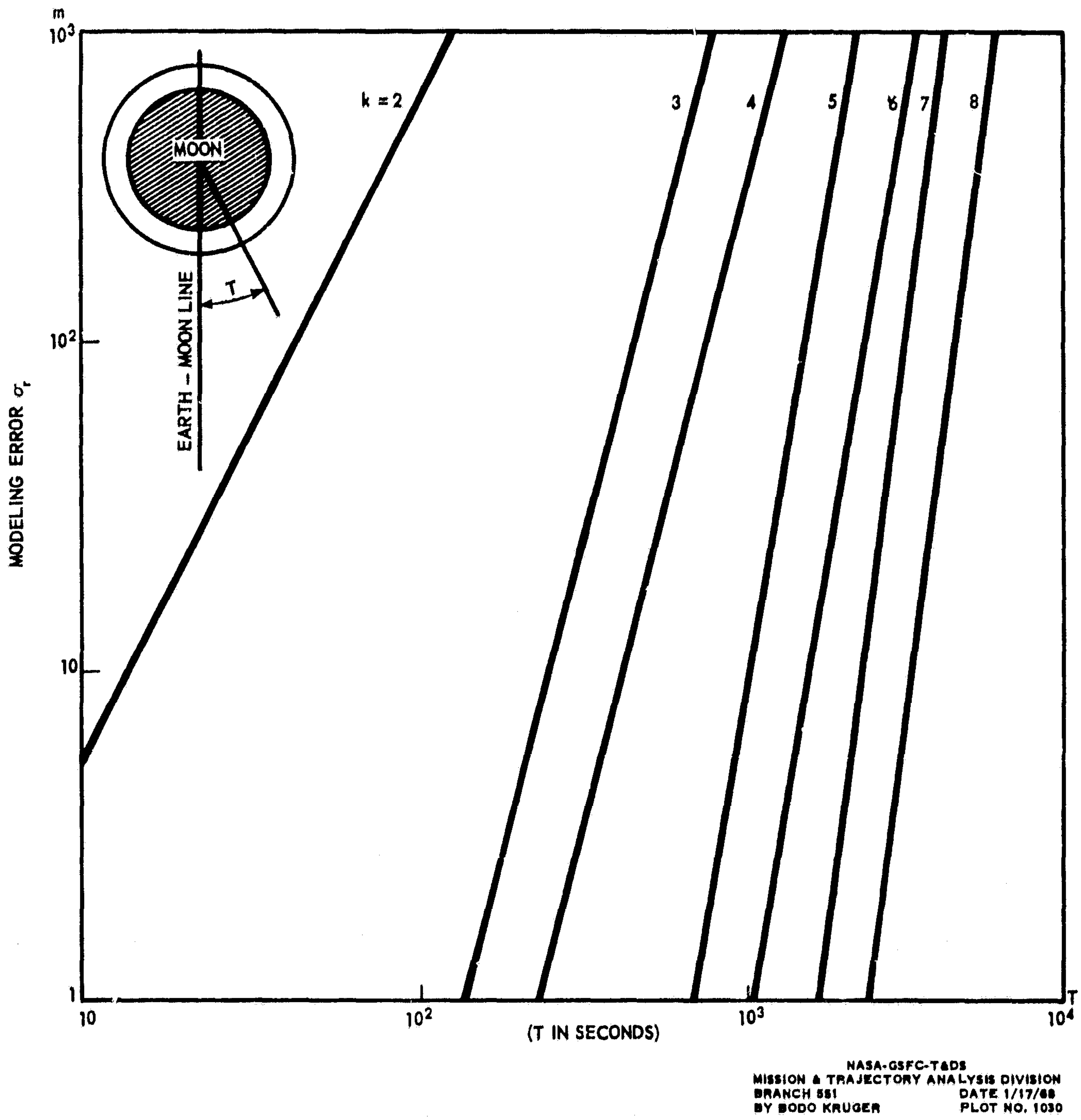

Figure 2. The standard deviation σ_r for the modeling error from a least squares polynomial fit, with k degrees of freedom, to range data from a lunar orbit. The interval of observation starts at the earth-moon line and is of duration T sec.

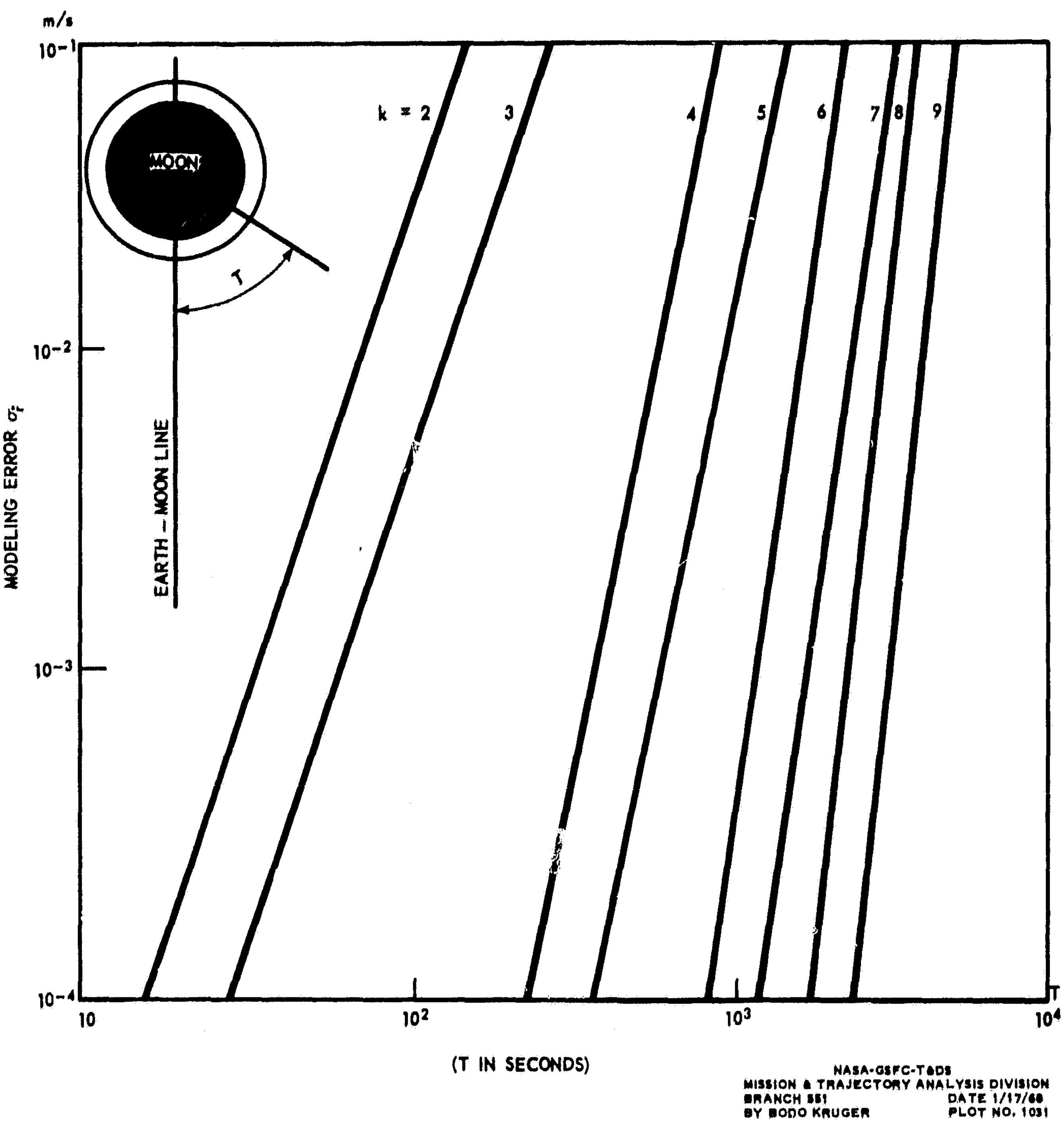

Figure 3. The standard deviation $\sigma_{\dot r}$ for the modeling error from a least squares polynomial fit, with k degrees of freedom, to range rate data from a lunar orbit. The interval of observation starts at the earth-moon line and is of duration T sec.

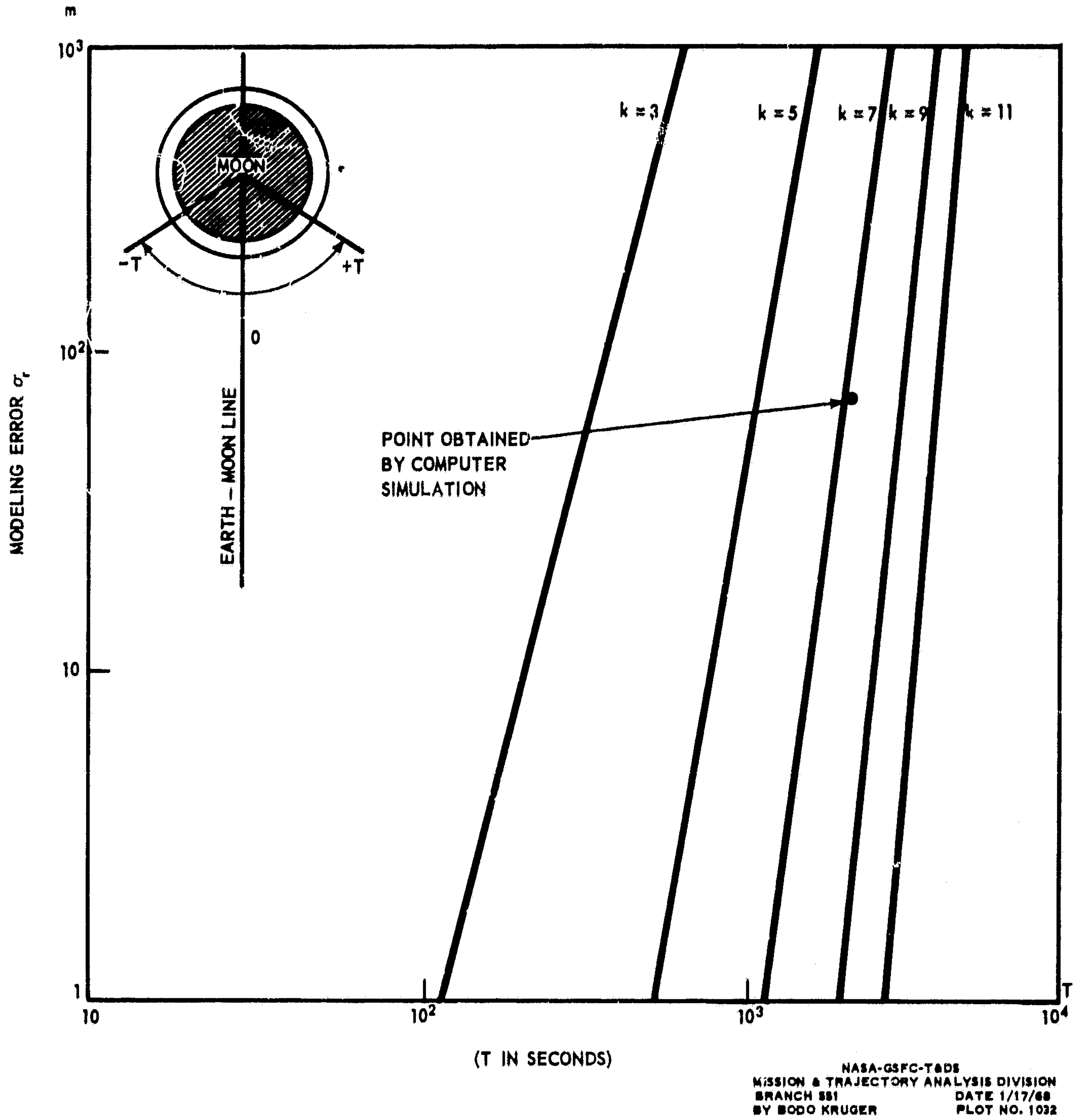

Figure 4. The standard deviation σ_r for the modeling error from a least squares polynomial fit, with k degrees of freedom, to range data from a lunar orbit. The interval of observation is from $-T$ to $+T$, symmetric around the earth-moon line.

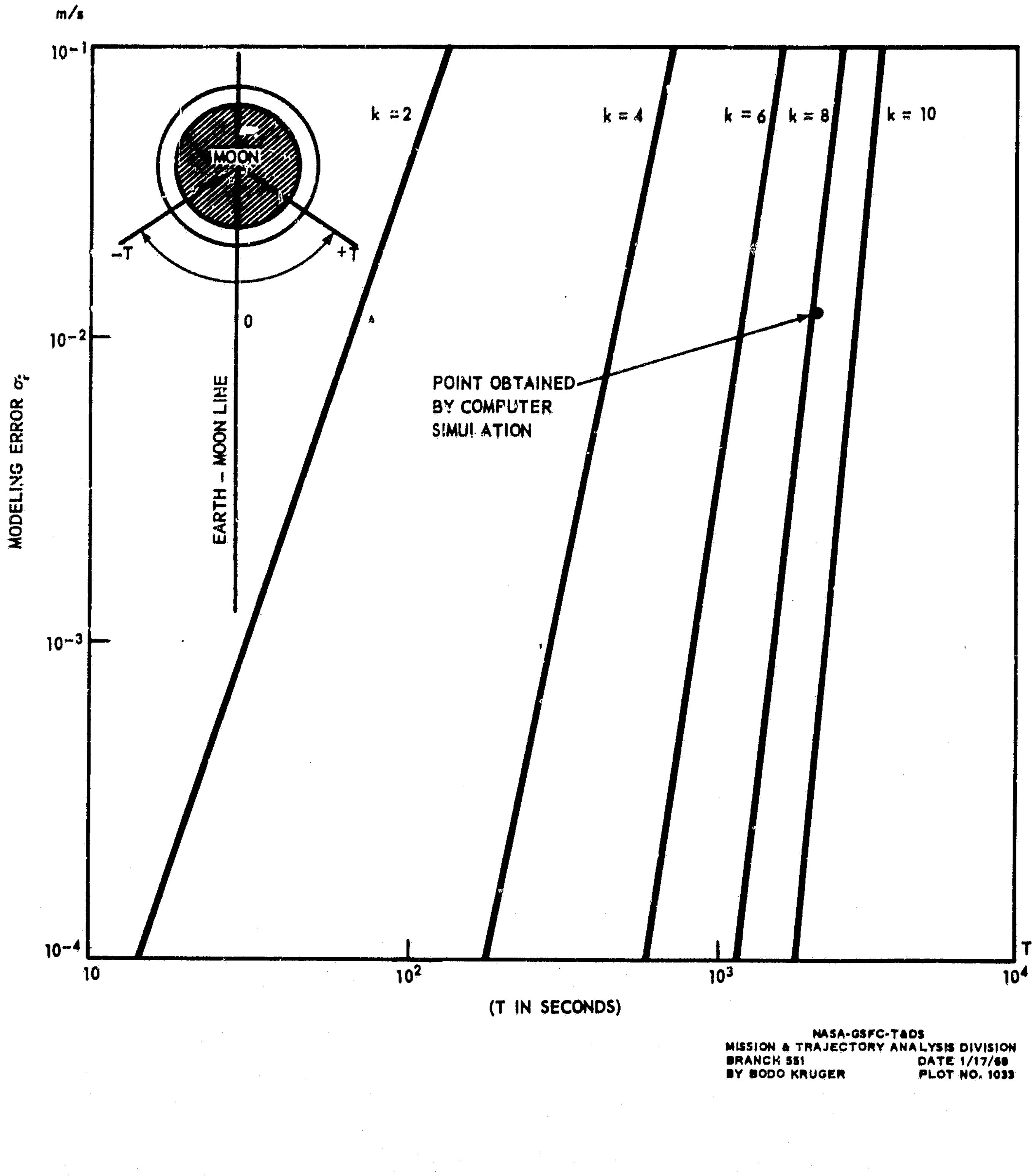

Figure 5. The standard deviation $\sigma_{\dot{r}}$ for the modeling error from a least squares polynomial fit, with k degrees of freedom, to range rate data from a lunar orbit. The interval of observation is from $-T$ to $+T$, symmetric around the earth-moon line.

4.0 THE LEAST SQUARES FITTED POLYNOMIAL AS LOW PASS FILTER

4.1 Bias and Noise

In this application of the least squares fitted polynomial the measured data points y_i are substituted by the LSP $\bar{y}_i$ in order to eliminate measurement noise. The LSP is thus used as a low pass filter. The problem in this application is to determine the difference δ_i between the true data point Y_i and the LSP $\bar{y}_i$.

$$\delta_i = Y_i - \bar{y}_i$$

If the measured points y_j are contaminated with noise ϵ_j , then

$$\bar{y}_i = -\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} (Y_j + \epsilon_j)$$

and from Equations (2.3) and (2.11)

$$\delta_i = \frac{(-1)^k}{|A|} \left\{ |i^k| \, b_k + |i^{k+1}| \, b_{k+1} + |i^{k+2}| \, b_{k+2} + \ldots \right\} \qquad (4.1)$$

$$+ \frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} \epsilon_j$$

The first term in this equation is "bias" due to the modeling error and the second term is the zero mean error due to measurement noise. The bias term will be denoted $\bar{\delta}_i$ and the variance of the noise $E\{\delta_i - \bar{\delta}_i\}^2$. In Appendix C it is shown that for large N, $\bar{\delta}_i$ may be written

$$\bar{\delta}_i = \frac{(-1)^k k!\, N^k}{2^k \prod_1^k (2j-1)} \left[P_k b_k + \frac{k+1}{2(2k+1)} \left\{ P_{k+1} + (2k+1) P_k \right\} N b_{k+1} \right.$$

$$\left. + \frac{(k+1)(k+2)}{4(2k+1)(2k+3)} \left\{ P_{k+2} + (2k+3) P_{k+1} + (k+2)(2k+1) P_k \right\} N^2 b_{k+2} + \cdots \right] \tag{4.2}$$

where P_k are the Legendre polynomials in u and

$$i = N \frac{u+1}{2} \tag{4.3}$$

If Y_i is expanded in a Taylor series an alternate expression is obtained using Equation (2.18) and (2.19)

$$\bar{\delta}_i = \frac{(-1)^k T^k}{2^k \prod_1^k (2j-1)} \left[P_k \cdot Y^{(k)} + \frac{T}{2(2k+1)} \left\{ P_{k+1} + (2k+1) P_k \right\} Y^{(k+1)} \right.$$

$$\left. + \frac{T^2}{4(2k+1)(2k+3)} \left\{ P_{k+3} + (2k+3) P_{k+1} + (k+2)(2k+1) P_k \right\} Y^{(k+2)} + \ldots \right] \tag{4.4}$$

where

T = the total length of the interval of observation

$Y^{(k)}$ = k^{th} derivation of Y

T is in the same units as the variable Y is expanded in.

If the ϵ_i are uncorrelated and have standard deviation σ_ϵ, then Equation (17) in Reference 1 is applicable and

$$E \left\{ \delta_i - \bar{\delta}_i \right\}^2 = \frac{\sigma_\epsilon^2}{N} \left\{ P_0^2 + 3 P_1^2 + \ldots + (2k+1) P_k^2 \right\} \tag{4.5}$$

where P_k are the Legendre Polynomials in u as before.

4.2 Data Compression

A form of data compression is to substitute the N original data points by one or more points from the LSP $\bar{y}_i$. The compressed data points are subject to the bias $\bar{\delta}_i$ as given by Equations (4.2) or (4.4). Therefore, $\bar{y}_i$ should be evaluated or interrogated at points where $\bar{\delta}_i$ is small.

From Equations (4.2) and (4.4) it is seen that the first term in the expansion of $\bar{\delta}_i$ is $P_k b_k$ or $P_k Y^{(k)}$. The effect of this term can be eliminated by interrogating $\bar{y}_i$ at a point where $P_k = 0$. For $k \geq 3$ we have several roots to choose from. In order to determine which root to use, the next term in the expansion of $\bar{\delta}_i$ is considered. This term is proportional to P_{k+1} as can be seen from Equations (4.2) and (4.4). We thus choose the root for which $|P_{k+1}|$ is smallest. The roots of P_k and $|P_{k+1}|$ for k values from 3 to 5 are listed below.

k = 3

$\lvert u \rvert$	P_3	$\lvert P_4 \rvert$
0	0	0.3750
0.7746	0	0.3000

k = 4

$\lvert u \rvert$	P_4	$\lvert P_5 \rvert$
0.3400	0	0.3294
0.8611	0	0.2444

k = 5

$\lvert u \rvert$	P_5	$\lvert P_6 \rvert$
0	0	0.3125
0.5385	0	0.2870
0.9062	0	0.2049

From the values of $|P_{k+1}|$ it is seen that the optimum root is the numerically largest root of $P_k = 0$.

A commonly used point of interrogation is the midpoint $u = 0$. It is clear from the above that this is a very poor choice for even k. For odd k some improvement is obtained by choosing the optimum root rather than $u = 0$. For $k = 3$ the improvement is a factor of 1.25 and for $k = 5$ the improvement factor is 1.53.

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APPENDIX A

THE EVALUATION OF

$$\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} j^{k+v} + i^{k+v}$$

$$\sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} j^{k+v} = \sum_{j=1}^{N} \begin{vmatrix} 0 & j^{k+v} & j^{k+v+1} & \cdots & j^{2k+v-1} \\ 1 & A_0 & A_1 & \cdots & A_{k-1} \\ i & A_1 & A_2 & \cdots & A_k \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ i^{k-1} & A_{k-1} & A_k & \cdots & A_{2k-2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & A_{k+v} & A_{k+v+1} & \cdots & A_{2k+v-1} \\ 1 & A_0 & A_1 & \cdots & A_{k-1} \\ i & A_1 & A_2 & \cdots & A_k \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ i^{k-1} & A_{k-1} & A_k & \cdots & A_{2k-2} \end{vmatrix} \tag{A-1}$$

because

$$\sum_{j=1}^{N} j^{v} = A_v$$

by definition. Thus

$$\frac{1}{|A|} \sum_{j=1}^{N} \begin{vmatrix} 0 & j \\ i & A \end{vmatrix} j^{k+v} + i^{k+v} =$$

$$\frac{1}{|A|} \begin{vmatrix} i^{k+v} & A_{k+v} & \cdot & \cdot & \cdot & \cdot & A_{2k+v-1} \\ 1 & A_0 & \cdot & \cdot & \cdot & \cdot & A_{k-1} \\ i & A_1 & & & & & A_k \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ i^{k-1} & A_{k-1} & \cdot & \cdot & \cdot & \cdot & A_{2k-2} \end{vmatrix}$$

$$= \frac{(-1)^k}{|A|} \begin{vmatrix} 1 & A_0 & \cdot & \cdot & \cdot & \cdot & A_{k-1} \\ i & A_1 & \cdot & \cdot & \cdot & \cdot & A_k \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ i^{k-1} & A_{k-1} & \cdot & \cdot & \cdot & \cdot & A_{2k-2} \\ i^{k+v} & A_{k+v} & \cdot & \cdot & \cdot & \cdot & A_{2k+v-1} \end{vmatrix} \qquad \text{(A-2)}$$

APPENDIX B

EVALUATION OF

$$\sum_{i=1}^{N} \left| i^{k+p} \right| \cdot \left| i^{k+q} \right|$$

$$\sum_{i=1}^{N} \left| i^{k+p} \right| \cdot \left| i^{k+q} \right| = \sum_{i=1}^{N} \begin{vmatrix} 1 & A_0 & \cdots & A_{k-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ i^{k-1} & A_{k-1} & \cdots & A_{2k-2} \\ i^{k+p} & A_{k+p} & \cdots & A_{2k-1+p} \end{vmatrix}$$

$$\cdot \begin{vmatrix} 1 & A_0 & \cdots & A_{k-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ i^{k-1} & A_{k-1} & \cdots & A_{2k-2} \\ i^{k+q} & A_{k+q} & \cdots & A_{2k+q} \end{vmatrix}$$

$$
= \sum_{i=1}^{N} \begin{vmatrix}
\begin{vmatrix} 1 & A_0 & \cdots \\ \vdots & \vdots & \\ i^{k-1} & A_{k-1} & \cdots \\ i^{k+p} & A_{k+p} & \cdots \end{vmatrix} & \begin{matrix} \\ \\ A_0 & \cdots & A_{k-1} \\ \\ \end{matrix} \\
\vdots & \begin{matrix} \vdots & & \vdots \end{matrix} \\
\begin{vmatrix} i^{k-1} & A_0 & \cdots \\ \vdots & \vdots & \\ i^{2k-2} & A_{k-1} & \cdots \\ i^{2k-1+p} & A_{k+p} & \cdots \end{vmatrix} & \begin{matrix} \\ \\ A_{k-1} & \cdots & A_{2k-2} \\ \\ \end{matrix} \\
\begin{vmatrix} i^{k+q} & A_0 & \cdots \\ \vdots & \vdots & \\ i^{2k-1+q} & A_{k-1} & \cdots \\ i^{2k+p+q} & A_{k+p} & \cdots \end{vmatrix} & \begin{matrix} \\ \\ A_{k+q} & \cdots & A_{2k-1+q} \\ \\ \end{matrix}
\end{vmatrix} \qquad \text{(B-1)}
$$

The determinants in the first column are all zero except the last one. Thus

$$\sum_{i=1}^{N} \left| i^{k+p} \right| \cdot \left| i^{k+q} \right| = |A| \cdot \begin{vmatrix} A_0 & \cdots & A_{k-1} & A_{k+q} \\ \vdots & & \vdots & \vdots \\ A_{k-1} & \cdots & A_{2k-2} & A_{2k-1+q} \\ A_{k+p} & \cdots & A_{2k-1+p} & A_{2k+p+q} \end{vmatrix} \tag{B-2}$$

For large N

$$A_v \approx \frac{N^{v+1}}{v+1}$$

so that

$$\begin{vmatrix} A_0 & \cdots & A_{k-1} & A_{k+q} \\ \vdots & & \vdots & \vdots \\ A_{k-1} & \cdots & A_{2k-2} & A_{2k-1+q} \\ A_{k+p} & \cdots & A_{2k-1+p} & A_{2k+p+1} \end{vmatrix} = N^{(1+k)^2+p+q} \begin{vmatrix} 1 & \cdots & \frac{1}{k} & \frac{1}{k+1+q} \\ \vdots & & \vdots & \vdots \\ \frac{1}{k} & \cdots & \frac{1}{2k-1} & \frac{1}{2k+q} \\ \frac{1}{k+1+p} & \cdots & \frac{1}{2k+p} & \frac{1}{2k+1+p+q} \end{vmatrix}$$

but

$$\begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k} & \frac{1}{k+1+q} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} & \frac{1}{k+2+q} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1} & \frac{1}{2k+q} \\ \frac{1}{k+1+p} & \frac{1}{k+2+p} & \cdots & \frac{1}{2k+p} & \frac{1}{2k+1+p+q} \end{vmatrix} =$$

$$\frac{(k+q)\,!}{(2k+q)\,!\;(2k+1+p+q)} \begin{vmatrix} k+1+q & \frac{k+1+q}{2} & \cdots & \frac{k+1+q}{k} \\ \frac{k+2+q}{2} & \frac{k+2+q}{3} & \cdots & \frac{k+2+q}{k+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{2k+q}{k} & \frac{2k+q}{k+1} & \cdots & \frac{2k+q}{2k-1} \\ \frac{2k+1+p+q}{k+1+p} & \frac{2k+1+p+q}{k+2+p} & \cdots & \frac{2k+1+p+q}{2k+p} \end{vmatrix}$$

$$
= \frac{(k+q)\,!}{(2k+q)\,!\,(2k+1+p+q)}
\begin{vmatrix}
\frac{(k+p)\,(k+q)}{k+1+p} & \frac{(k+p)\,(k-1+q)}{2(k+2+p)} & \cdots & \frac{(k+p)\,(1+q)}{k\,(2k+p)} \\
\frac{(k-1+p)\,(k+q)}{2\,(k+1+p)} & \frac{(k-1+p)\,(k-1+q)}{3\,(k+2+p)} & \cdots & \frac{(k-1+p)\,(1+q)}{(k+1)\,(2k+p)} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\frac{(1+p)\,(1+q)}{k\,(k+1+p)} & \frac{(1+p)\,(k-1+q)}{(k+1)\,(k+2+p)} & \cdots & \frac{(1+p)\,(1+q)}{(2k-1)\,(2k+p)}
\end{vmatrix}
$$

$$
= \frac{[(k+p)\,!]^2\;[(k+q)]^2}{p!\;q!\;(2k+p)!\;(2k+q)!\;(2k+1+p+q)}
\begin{vmatrix}
1 & \frac{1}{2} & \cdots & \frac{1}{k} \\
\frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1}
\end{vmatrix}
$$

(B-3)

In the first step the rows are multiplied by factors so that "one's" are obtained in the last column and in the second step the bottom row is subtracted from the other rows. In Reference 1, Equation (C-7) it is shown that for large N

$$|A| = N^{k^2} \begin{vmatrix} 1 & \cdots & \frac{1}{k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{1}{k} & \cdots & \frac{1}{2k-1} \end{vmatrix}$$

so that for large N

$$\sum_{i=1}^{N} \left| i^{k+p} \right| \cdot \left| i^{k+q} \right| = N^{2k+1+p+q} \frac{[(k+p)!]^2 \; [(k+q)!]^2}{p!\; q!\; (2k+p)!\; (2k+q)!\; (2k+1+p+q)} \; |A|^2 \tag{B-4}$$

APPENDIX C

EVALUATION OF

$$\frac{|i^{k+v}|}{|\Lambda|}$$

With $v = 0$ and the substitution

$$i = N \frac{u+1}{2}$$

we obtain for large N

$$|i^k| = \begin{vmatrix} 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \frac{u+1}{2} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \left(\frac{u+1}{2}\right)^2 & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \left(\frac{u+1}{2}\right)^k & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k} \end{vmatrix} N^{\frac{1}{2}(k+1)}$$

But

$$\left(\frac{u+1}{2}\right)^k = \frac{k!}{2^k \prod_1^k (2j-1)} \left\{P_k + (2k-1)\, P_{k-1} + k\,(2k-3)\, P_{k-2} + \ldots\right\}$$

$$= \alpha_k P_k + \alpha_{k-1} P_{k-1} + \alpha_{k-2} P_{k-1} \qquad \text{(C-1)}$$

where $P_k = P_k(u)$ are the Legendre Polynomials. Thus

$$|i^k| = \begin{vmatrix} P_0 & 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \frac{1}{2} P_1 + \frac{1}{2} P_0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \frac{1}{6} P_2 + \frac{1}{2} P_1 + \frac{1}{3} P_0 & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \alpha_k P_k + \alpha_{k-1} P_{k-1} + \alpha_{k-2} P_{k-2} + \ldots & \frac{1}{k+1} & \frac{1}{k+2} & \cdots & \frac{1}{2k} \end{vmatrix} N^{k(k+1)} \qquad \text{(C-2)}$$

P_0 may be eliminated by subtracting the top row from the others, yielding

$$
|i^k| = \begin{vmatrix} P_0 & 1 & \frac{1}{2} & \cdots \\ \frac{1}{2} P_1 & 0 & \frac{1}{12} & \cdots \\ \frac{1}{6} P_2 + \frac{1}{2} P_1 & 0 & \frac{1}{12} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \alpha_k P_k + \alpha_{k-1} P_{k-1} + \alpha_{k-2} P_{k-2} & 0 & & \end{vmatrix} N^{k(k+1)} \tag{C-3}
$$

Repeating the process

$$
|i^k| = \begin{vmatrix} \frac{2(2k-1)}{k} \alpha_k P_{k-1} & |K| \\ \alpha_k P_k & 0 \end{vmatrix} \tag{C-4}
$$

where

$$
|K| = \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{k} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{1}{k} & \frac{1}{k+1} & \cdots & \frac{1}{2k-1} \end{vmatrix} \tag{C-5}
$$

But for large N

$$|A| = |K|\ N^{2k}$$

so that

$$\frac{|i^k|}{|A|} = -\frac{k!}{2^k \prod_1^k (2j-1)} P_k N^k \tag{C-6}$$

For $v = 1$ we obtain in the same manner

$$\frac{|i^{k+1}|}{|A|} = -\frac{(k+1)!}{2^{k+1} \prod_1^{k+1} (2j-1)} \left\{P_{k+1} + (2k+1) P_k\right\} N^{k+1} \tag{C-7}$$

and for $v = 2$

$$\frac{|i^{k+2}|}{|A|} = -\frac{(k+2)!}{2^{k+2} \prod^{k+2} (2j-1)} \left\{P_{k+2} + (2k+3) P_{k+1} + (k+2)(2k+1) P_k\right\} N^{k+2} \tag{C-8}$$